ASYMPTOTIC BEHAVIOR OF NONLINEAR SCHRÖDINGER SYSTEMS WITH LINEAR COUPLING

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ABSTRACT. A system of two coupled nonlinear Schrödinger equations is treated. In addition, a linear coupling which models an external driven field described by the Rabi frequency is considered. Asymptotics for large Rabi frequency are carried out. Convergence in the appropriate Strichartz space is proven. As a consequence, the global existence of the limiting system gives a criterion for the long time behavior of the original system.

1. INTRODUCTION

We consider the following system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t\psi_1 = -\frac{1}{2}\Delta\psi_1 + \frac{\gamma^2}{2}|x|^2\psi_1 + \beta_{11}|\psi_1|^2\psi_1 + \beta_{12}|\psi_2|^2\psi_1 + \lambda\psi_2 \\ i\partial_t\psi_2 = -\frac{1}{2}\Delta\psi_2 + \frac{\gamma^2}{2}|x|^2\psi_2 + \beta_{12}|\psi_1|^2\psi_2 + \beta_{22}|\psi_2|^2\psi_2 + \lambda\psi_1, \\ \psi_1(0) = \psi_{1,0}, \quad \psi_2(0) = \psi_{2,0} \end{cases} \tag{1}$$

with $x \in \mathbb{R}^N$ and $N \leq 3$. The parameters are the magnetic trap strength $\gamma > 0$, the scattering lengths $\beta_{11}, \beta_{12}, \beta_{22} \in \mathbb{R}$ and the external driven field constant $\lambda \in \mathbb{R}$, called Rabi frequency.

The energy and the mass associated to system (1) are the following

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla\psi_1|^2 + \frac{1}{2}|\nabla\psi_2|^2 + \frac{\gamma^2}{2}|x|^2(|\psi_1|^2 + |\psi_2|^2) \\ &\qquad + \frac{\beta_{11}}{2}|\psi_1|^4 + \beta_{12}|\psi_1|^2|\psi_2|^2 + \frac{\beta_{22}}{2}|\psi_2|^4 + 2\lambda\Re(\psi_1^*\psi_2)\Big)(x,t)dx, \\ M(t) &= \int_{\mathbb{R}^N} \Big(|\psi_1|^2 + |\psi_2|^2\Big)(x,t)dx, \end{aligned} \tag{2}$$

and are conserved along the flow of solutions to system (1).

The above system models a mixture of Bose-Einstein condensates consisting of two different hyperfine states of Rubidium atoms confined in the same harmonic trap [3]. The first experiment concerning the binary Bose-Einstein condensate was performed at JILA with $|F = 2, m_f = 2\rangle$ and $|1, -1\rangle$ spin states of ^{87}Rb [26]. By applying a weak magnetic (driven) field with Rabi frequency λ, the two components were coupled in the overlap region. This coupling realizes a Josephson-type junction and gives rise to nonlinear oscillations in the

Date: March 27, 2013.

relative populations [37]. The effect of the Rabi term was examined in several papers [13, 27, 28, 33]. In [13, 28] it is shown that the system may be transformed into a system without the Rabi term given that all the interaction strengths are the same (the β_{ij}'s are equal). Nistazakis et. al. [28] showed that due to the Rabi frequency it is possible to transfer the wave function of one component to the other.

An interesting question in this framework regards the dynamics of a two-component Bose-Einstein condensate with mixed attractive/repulsive nonlinearities. Indeed, depending on the intra- and inter-species scattering lengths, the system will either reach a stable configuration, or there will be occurrence of a singular dynamics. It is a standard procedure in physical experiments for single-component Bose-Einstein condensates, to use a Feshbach-resonance technique in order to tune the scattering length and to consequently stabilize the condensate. In mathematical language this is translated by introducing a time-oscillating coefficient in front of the nonlinearity in the Gross-Pitaevskii equation [1, 20, 31]. This stabilization method attracted a wide interest in physical literature and it is studied extensively from the experimental, numerical and theoretical point of view. However, the use of Feshbach resonance would imply the strength of the applied magnetic field to oscillate at frequencies much higher than the characteristic frequencies of the system. To overcome this difficulty, Saito et al [32] proposed to use two hyperfine states with different scattering length. In such a system the Rabi oscillation between the states will cause an effective oscillation of the scattering lengths and stabilize the system; this is performed by irradiating the system with a constant electromagnetic wave. Our purpose in this paper is to give a rigorous analysis for such systems. We investigate the dynamics of a two-component nonlinear Schrdinger system with focusing and defocusing nonlinearities driven by an external field with Rabi frequency λ. A considerable amount of mathematical literature is dedicated to the study of long time behavior of nonlinear Schrdinger systems (see references below), however the case of systems with a linear coupling term coming from the Rabi frequency is poorly understood. Numerical experiments performed in [17] suggest the fact that the Rabi frequency may affect the long time behavior of the system. Therefore here we address to the question whether the presence of an external driven field can prevent blow-up in the presence of focusing nonlinearities. More in particular we investigate the asymptotic behavior of the system when the Rabi frequency becomes large, and we rigorously study the limit as $|\lambda| \to \infty$. We show convergence in Strichartz spaces of solutions to system (1) towards solutions to

$$\begin{cases} i\partial_t u_1 = -\frac{1}{2}\Delta u_1 + \frac{\gamma^2}{2}|x|^2 u_1 + \sigma_1 |u_1|^2 u_1 + \sigma_2 |u_2|^2 u_1 \\ i\partial_t u_2 = -\frac{1}{2}\Delta u_2 + \frac{\gamma^2}{2}|x|^2 u_2 + \sigma_1 |u_2|^2 u_2 + \sigma_2 |u_1|^2 u_2, \end{cases}$$

with $\sigma_1 = (\beta_{11} + 2\beta_{12} + \beta_{22})/2, \sigma_2 = (\beta_{11} + \beta_{22})/2$, when $|\lambda| \to \infty$.

Moreover, the solution of the system (1) is proven to exist on a time interval strictly less than the maximal existence time of the limiting system. This implies in the case of global existence of the limiting solution, that the solution of the original system exists on finite time $[0, T]$, with arbitrarily large T. Therefore, at least for large λ we expect the original

nonlinear system to behave like the limiting system, which is nothing but a well-known nonlinear Schrödinger system.

Here we give a short overview on the papers dealing with global existence/blow-up of the system. The following Schrödinger system without external potential and with power-type nonlinearities was extensively treated in the literature:

$$\begin{cases} i\partial_t\psi_1 = -\dfrac{1}{2}\Delta\psi_1 + \Big(\beta_{11}|\psi_1|^{2p} + \beta_{12}|\psi_1|^{p-1}|\psi_2|^{p+1}\Big)\psi_1 \\ i\partial_t\psi_2 = -\dfrac{1}{2}\Delta\psi_2 + \Big(\beta_{12}|\psi_1|^{p+1}|\psi_1|^{p-1} + \beta_{22}|\psi_2|^{2p}\Big)\psi_2. \end{cases}$$

Fanelli and Montefusco [14] deal with this system with $\beta_{12} < 0$, and $\beta_{11} = \beta_{22} = -1$. Using the existence of ground-state solutions they establish a vector-valued Gagliardo–Nirenberg inequality, which also gives the mass threshold for global existence of solutions to the system in the mass critical case $p = 2/N$. Li, Wu and Lai [21] improve partially the arguments of Fanelli and Montefusco in the critical and super critical case $p \geq 2/N$. They obtain a new blow-up threshold in terms of the ground state. The same results for more general coefficients of the nonlinearities are established by Ma and Zhao [25], by using the sharp vector-valued Gagliardo–Nirenberg inequality they derived previously in [24]. The planar case $N = 2$ is treated by Ma and Schulze [23]. Lin and Wei [22] prove simultaneous blow-up of the two components by studying the associated self-similar solutions for $p = 1$. Song [34] considers the system with more general power-type nonlinearities and establishes the criterion for blow-up and global existence in the critical case. The blow-up profile is analyzed by Chen and Guo [11]. In the critical case $p = 2/N$ they show that the solution blows up like the Dirac δ distribution. Moreover, they carry out the result for the system with harmonic potential by using the Lens Transform [7, 35]. Note that in the case $p = 1$ we recover (1).

Among the papers dealing with nonlinear Schrödinger systems with an external potential we mention [12, 30, 38]. In [12] the authors consider (1) with all $\beta_{ij} = -1$ and derive a sharp criterion for the energy norm for blow-up/global existence using the existence of the ground state solution of the scalar equation without potential. Prytula et al. [30] study a system of M equations and prove a sufficient condition for blow-up depending on the coefficients β_{ij}. The two-dimensional case $N = 2$ for a system (1) with harmonic potential was considered by Zhongxue and Zuhan [38]. Here the ground state solution of the system is related to the threshold condition for blow-up.

There are only few papers considering the system with the Rabi frequency. Bao and Cai [4] treat the stationary case, and prove existence and uniqueness of ground states. In this framework they also perform asymptotics for $|\lambda| \to \infty$ and obtain the limiting behavior of the ground states. The solitary wave solutions of a two-component system when all the physical parameters are spatially dependent is studied in [5]. Zhongxue and Zuhan [39] prove finite-time blow-up of solutions for initial data greater than a threshold when all nonlinearities are focusing. The results seems to be valid in the two-dimensional case only. The nonlinear Schrödinger system of two equations with an external driven field and with nonlinearities which may not be cubic is analyzed in [17]. The authors derive

sufficient conditions for blow-up and global existence. A semi-implicit formula shows the mass transport between the two components.
In [10] Cazenave and Scialom study rigorously the asymptotic behavior of a nonlinear Schrödinger equation with a time-periodic coefficient in front of the nonlinearity. Fang and Han [15] prove the convergence in the case of an energy critical nonlinearity, thus the power of the nonlinearity is $4/(N-2)$.
This paper is organized as follows: in Section 2 we introduce the main notations and tools used throughout the paper. We furthermore study system (1), establishing the cases when there is global existence of solutions or possible occurrence of finite time blow-up, according to the choice of the coefficients β_{ij}'s. Then in Section 3 we transform (1) in a suitable way, in order to study the system when the Rabi frequency λ becomes large. We then write down the limiting system, and in Section 4 we prove the rigorous convergence result, namely Theorem 11, which is the main result of our paper. As the reader will see in Section 4 the proof of Theorem 11 follows the same steps as the paper by Cazenave and Scialom [10].

2. Notation and preliminary results

In what follows C will denote a generic constant greater than 1, which may possibly change from line to line. With $\Re$ and $\Im$ we denote the real and imaginary part of a complex number, respectively. By z^* we denote the complex conjugate of z. The scalar product between two vectors v, v' will be denoted by $\langle v, v'\rangle$.
Since we are dealing with two-component Schrödinger systems, we shall indicate with capital letters the two-dimensional vector fields describing the wave-function of a two-component quantum system. For example we shall write $\Psi^t = (\psi_1, \psi_2)$, or $\Psi_0^t = (\psi_{1,0}, \psi_{2,0})$, $\Psi^* = (\psi_1^*, \psi_2^*)$ and so on. Consequently, we also denote

$$|\Psi|^2 = |\psi_1|^2 + |\psi_2|^2.$$

In this way we may write system (1) in the following compact form

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi + \frac{\gamma^2}{2}|x|^2\Psi + \tilde{B}[\Psi]\Psi + A\Psi, \tag{3}$$

where $A, \tilde{B}[\Psi]$ are the two matrices defined by

$$A = \begin{pmatrix} & \lambda \\ \lambda & \end{pmatrix}, \tag{4}$$

$$\tilde{B}[\Psi] = \begin{pmatrix} \beta_{11}|\psi_1|^2 & \beta_{12}\psi_1\psi_2^* \\ \beta_{12}\psi_1^*\psi_2 & \beta_{22}|\psi_2|^2 \end{pmatrix}. \tag{5}$$

The energy and mass may be written

$$E(t) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla\Psi|^2 + \frac{\gamma^2}{2}|x|^2|\Psi|^2 + \frac{1}{2}\Psi^*\tilde{B}[\Psi]\Psi + 2\lambda\Re(\psi_1^*\psi_2)\Big)(x,t)dx. \tag{6}$$

$$M(t) = \int_{\mathbb{R}^N} |\Psi(x,t)|^2 dx. \tag{7}$$

We shall denote by $L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N)$, the usual Lebesgue and Sobolev spaces, respectively. We shall also make use of the mixed space-time Lebesgue (or Sobolev) spaces, so that for example $L^q(I; L^r(\mathbb{R}^N))$ denote the space of those functions f having the following norm finite,

$$\|f\|_{L^q(I;L^r(\mathbb{R}^N))} := \left(\int_I \left(\int_{\mathbb{R}^N} |f(t,x)|^r dx\right)^{q/r} dt\right)^{\frac{1}{r}}.$$

We often shorten notation $L^q_t L^r_x(I \times \mathbb{R}^N) = L^q(I; L^r(\mathbb{R}^N))$ or $L^q_t L^r_x$ if there is no source of ambiguity.

We are interested in studying system (1) in the *energy space*, which is defined by

$$\Sigma(\mathbb{R}^N) := \{f \in H^1(\mathbb{R}^N);\ |\cdot| f \in L^2(\mathbb{R}^N)\}.$$

It is straightforward to see that, for any $\Psi \in \Sigma(\mathbb{R}^N)$, the energy in (2) is finite.

Let us consider the Hamiltonian with confining potential, $H = -\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2$, the associated propagator is $S_0(t) := e^{-itH}$. It is unitary on $L^2(\mathbb{R}^N)$ and from Melher's formula (see for example [8, 29]) one can see that it satisfies a dispersive estimate for short times, namely

$$\|S_0(t)f\|_{L^\infty(\mathbb{R}^N)} \lesssim |t|^{-\frac{N}{2}} \|f\|_{L^1(\mathbb{R}^N)}, \quad |t| \leq \delta, \tag{8}$$

for some small $\delta > 0$ depending on γ.

Furthermore, in what follows we also need the following commutation formulas for H:

$$[\nabla, H] = \gamma^2 x, \quad [x, H] = \nabla. \tag{9}$$

Even if (8) holds only for short times, we know the group $S_0(t)$ enjoys the same Strichartz estimates as the free Schrödinger propagator without confining potential, but only locally in time, i.e. the constants appearing in the inequalities below depend on the length of the time interval. Here we state the results we need, for further details we address the reader to [8], Section 2.

Definition 1. *We say (q, r) is an* admissible pair *if $2 \leq r \leq \frac{2N}{N-2}$ ($2 \leq r \leq \infty$ for $N = 1$, $2 \leq r < \infty$ for $N = 2$), and*

$$\frac{1}{q} = \frac{N}{2}\left(\frac{1}{2} - \frac{1}{r}\right). \tag{10}$$

Proposition 2. *Let $(q, r), (\tilde{q}, \tilde{r})$ be two arbitrary admissible pairs. Then, for any compact time interval, we have*

$$\begin{aligned}\|S_0(t)f\|_{L^q_t L^r_x(I\times\mathbb{R}^N)} \leq& C(|I|, r)\|f\|_{L^2(\mathbb{R}^N)},\\ \left\|\int_0^t S_0(t-s)F(s)ds\right\|_{L^q_t L^r_x(I\times\mathbb{R}^N)} \leq& C(|I|, r, \tilde{r})\|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(I\times\mathbb{R}^N)}.\end{aligned}$$

The estimates in Proposition 2 will serve us to show local well-posedness of (1) in $\Sigma(\mathbb{R}^N)$. As we already said in the Introduction, in order to study the asymptotic behavior of (1), we first transform the system into a similar one, see (22), and then we perform the limit, obtaining (27). All those systems are quite similar and at a local level they can be studied

in the same way. For this reason here we write a more general local well-posedness result, which may be applied to all those systems involved in this paper. The result in Proposition below is quite standard; its proof consists in just adapting the method by Kato introduced in [18] for NLS equations, to the case of systems and for Hamiltonians with a confining potential. As it is already well explained in [9] (see Chapter 4 and also Remark 3.3.12), the known theory for nonlinear Schrödinger equations is easily extended to systems. In [9], [18] the presence of a confining potential is not considered, however this is only a minor modification for the local well-posedness framework, thanks to Strichartz estimates stated in Proposition 2.

Proposition 3. *Let us consider the following Cauchy problem*

$$\begin{cases} i\partial_t\Psi = -\frac{1}{2}\Delta\Psi + \frac{\gamma^2}{2}|x|^2\Psi + \mathcal{N}(\Psi), \\ \Psi(0) = \Psi_0, \end{cases} \tag{11}$$

where the unknown Ψ *is a complex vector field. Let* $\mathcal{N} \in \mathcal{C}(\mathbb{C}^2;\mathbb{C}^2)$ *be such that* $\mathcal{N}(0) = 0$, *and* $\mathcal{N}(\Psi) = \mathcal{N}_1(\Psi) + \mathcal{N}(\Psi_2)$, *where* $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{C}(\mathbb{C}^2;\mathbb{C}^2)$ *satisfy*

- $|\mathcal{N}_1(\Psi) - \mathcal{N}_2(\tilde\Psi)| \leq C|\Psi - \tilde\Psi|$;
- $|\mathcal{N}_1(\Psi) - \mathcal{N}_2(\tilde\Psi)| \leq C(|\Psi|^2 + |\tilde\Psi|^2)|\Psi - \tilde\Psi|$,

for all $\Psi, \tilde\Psi \in \mathbb{C}^2$.
For any $\Psi_0 \in \Sigma(\mathbb{R}^N)$, *there exist* $\delta = \delta(\|\Psi_0\|_\Sigma) > 0$ *and a unique solution* $\Psi \in \mathcal{C}([0,\delta];\Sigma(\mathbb{R}^N))$ *to* (11). *Moreover we have*

$$\|\Psi\|_{L^\infty([0,\delta];\Sigma(\mathbb{R}^N))} \leq 2C\|\Psi_0\|_{\Sigma(\mathbb{R}^N)}. \tag{12}$$

Furthermore, the solution Ψ *can be extended to a maximal interval* $[0, T_{max})$, *and the blow-up alternative holds true, namely if* $T_{max} < \infty$, *then*

$$\lim_{t\to T_{max}} \|\nabla\Psi(t)\|_{L^2(\mathbb{R}^N)} = \infty;$$

Finally, for any $0 < T < T_{max}$ *and any admissible pair* (q, r), *we have* $\Psi, \nabla\Psi, |\cdot|\Psi \in L^q([0,T];L^r(\mathbb{R}^N))$.

Proof. Let us define the space

$$\begin{aligned} K := \{\Psi \text{ s.t. } \Psi, \nabla\Psi, |\cdot|\Psi \in L^\infty_t L^2_x([0,\delta]\times\mathbb{R}^N)) \cap L^{8/N}_t L^4_x([0,\delta]\times\mathbb{R}^N)), \\ \|\Psi\|_{L^\infty([0,\delta];\Sigma(\mathbb{R}^N))} + \left\| \begin{pmatrix} 1 \\ \nabla \\ |\cdot| \end{pmatrix} \Psi \right\|_{L^{8/N}_t L^4_x([0,\delta]\times\mathbb{R}^N)} \leq M\}, \end{aligned}$$

where $M, \delta > 0$ will be chosen later, endowed with the distance

$$d(\Psi, \tilde\Psi) := \|\Psi - \tilde\Psi\|_{L^\infty_t L^2_x([0,\delta]\times\mathbb{R}^N)} + \|\Psi - \tilde\Psi\|_{L^{8/N}_t L^4_x([0,\delta]\times\mathbb{R}^N)}.$$

By the hypotheses on $\mathcal{N}(\Psi)$ we have

$$\begin{aligned}
\|\mathcal{N}_1(\Psi)-\mathcal{N}_1(\tilde{\Psi})\|_{L^2(\mathbb{R}^N)} \leq & C\|\Psi-\tilde{\Psi}\|_{L^2(\mathbb{R}^N)}\\
\|\mathcal{N}_2(\Psi)-\mathcal{N}_2(\tilde{\Psi})\|_{L^{4/3}(\mathbb{R}^N)} \leq & C(\|\Psi\|^2_{L^4(\mathbb{R}^N)}+\|\tilde{\Psi}\|^2_{L^4(\mathbb{R}^N)})\|\Psi-\tilde{\Psi}\|_{L^4(\mathbb{R}^N)}\\
\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\mathcal{N}_1(\Psi)\|_{L^2(\mathbb{R}^N)} \leq & C\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\Psi\|_{L^2(\mathbb{R}^N)}\\
\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\mathcal{N}_2(\Psi)\|_{L^{4/3}(\mathbb{R}^N)} \leq & C\|\Psi\|^2_{L^4(\mathbb{R}^N)}\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\Psi\|_{L^4(\mathbb{R}^N)}.
\end{aligned}$$

Let us now consider the operator

$$G[\Psi]:=S_0(t)\Psi_0-i\int_0^t S_0(t-s)\mathcal{N}(\Psi)(s)ds.$$

By using the commutation rules for the Hamiltonian H and Strichartz estimates, we have

$$\begin{aligned}
&\|G[\Psi]\|_{L_t^\infty\Sigma_x}+\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}G[\Psi]\|_{L_t^{8/N}L_x^4([0,\delta]\times\mathbb{R}^N)}\\
&\qquad\leq C\|\Psi_0\|_\Sigma+C\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\mathcal{N}_1(\Psi)\|_{L_t^1L_x^2}+C\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\mathcal{N}_2(\Psi)\|_{L_t^{\frac{8}{8-N}}L_x^{4/3}}\\
&\qquad\leq C\|\Psi_0\|_\Sigma+C\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\Psi\|_{L_t^1L_x^2}+C\|\Psi\|^2_{L_t^\infty L_x^4}\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\Psi\|_{L_t^{\frac{8}{8-N}}L_x^4}.
\end{aligned}$$

Analogously we obtain

$$\begin{aligned}
\|G[\Psi]-G[\tilde{\Psi}]\|_{L_t^\infty L_x^2}+\|G[\Psi]-G[\tilde{\Psi}]\|_{L_t^{8/N}L_x^4} &\leq C\|\mathcal{N}_1(\Psi)-\mathcal{N}_1(\tilde{\Psi})\|_{L_t^1L_x^2}+C\|\mathcal{N}_2(\Psi)-\mathcal{N}_2(\tilde{\Psi})\|_{L_t^{\frac{8}{8-N}}L_x^{4/3}}\\
&\leq C\|\Psi-\tilde{\Psi}\|_{L_t^1L_x^2}+C(\|\Psi\|^2_{L_t^\infty L_x^4}+\|\tilde{\Psi}\|^2_{L_t^\infty L_x^4})\|\Psi-\tilde{\Psi}\|_{L_t^{\frac{8}{8-N}}L_x^4}.
\end{aligned}$$

By the Sobolev embedding $H^1\hookrightarrow L^4$ and by using Hölder's inequality in time in the previous expressions, we have

$$\begin{aligned}
\|G[\Psi]\|_{L_t^\infty\Sigma_x([0,\delta]\times\mathbb{R}^N)}+\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}G[\Psi]\|_{L_t^{8/N}L_x^4([0,\delta]\times\mathbb{R}^N)} \leq & C\|\Psi_0\|_\Sigma+C(\delta+\delta^{\frac{8-2N}{8}}M^2)M\\
\|G[\Psi_1]-G[\Psi_2]\|_{L_t^{8/N}L_x^4([0,\delta]\times\mathbb{R}^N)} \leq & C(\delta+\delta^{\frac{8-2N}{8}}M^2)d(\Psi,\tilde{\Psi}).
\end{aligned}$$

Now, we choose M, δ such that

$$\begin{aligned} C\|\Psi_0\|_{\Sigma(\mathbb{R}^N)} =& \frac{M}{2} \\ C(\delta + \delta^{\frac{8-2N}{8}} M^2) \leq & \frac{1}{2}, \end{aligned}$$

so that we have

$$\begin{aligned} &\|G[\Psi]\|_{L^\infty([0,\delta];\Sigma(\mathbb{R}^N))} + \|\begin{pmatrix} 1 \\ \nabla \\ |\cdot| \end{pmatrix} G[\Psi]\|_{L_t^{8/N} L_x^4([0,\delta]\times\mathbb{R}^N)} \leq M \\ &d(G[\Psi], G[\tilde{\Psi}]) \leq \frac{1}{2} d(\Psi, \tilde{\Psi}). \end{aligned} \tag{13}$$

This implies G is a contraction in K, therefore there exists a unique $\Psi \in K$ such that

$$\Psi(t) = G[\Psi](t) = S_0(t)\Psi_0 - i\int_0^t S_0(t-s)\mathcal{N}(\Psi)(s)ds.$$

Hence $\Psi \in \mathcal{C}([0,\delta];\Sigma(\mathbb{R}^N)) \cap L^{8/N}([0,\delta];L^4(\mathbb{R}^N))$ is a solution to (11) in $[0,\delta]$. From (13) we also see

$$\|\Psi\|_{L^\infty([0,\delta];\Sigma(\mathbb{R}^N))} \leq M = 2C\|\Psi_0\|_{\Sigma(\mathbb{R}^N)},$$

which proves (12). Analogously, by Strichartz estimates we also have

$$\|\begin{pmatrix} 1 \\ \nabla \\ |\cdot| \end{pmatrix} \Psi\|_{L_t^q L_x^r([0,\delta]\times\mathbb{R}^N)} \leq C\|\Psi_0\|_\Sigma + C\delta^{\frac{8-2N}{N}}(1+M^2)M \leq M.$$

Furthermore, from the proof in the fixed point argument above we also infer that we may extend the solution as long as the L^2-norm of the gradient of the solution remains bounded, hence the blow-up alternative holds true. This implies we can extend the solution to a maximal interval $[0, T_{max})$, and moreover for any $0 < T < T_{max}$, (q,r) admissible pair, we have

$$\|\begin{pmatrix} 1 \\ \nabla \\ |\cdot| \end{pmatrix} \Psi\|_{L_t^q L_x^r([0,T]\times\mathbb{R}^N)} < \infty.$$

□

Corollary 4. *The system* (1) *is locally well-posed in* $\Sigma(\mathbb{R}^N)$. *Furthermore the energy and the mass defined in* (2) *are conserved in* $[0, T_{max})$.

A natural question which arises at this point is to see whether the solution in Corollary above is global or there is possible occurrence of blow-up in finite time.
For the case of a single cubic nonlinear Schrödinger equation the picture is complete.

- If the nonlinearity is defocusing (i.e. its coefficient is positive) then, by using the conservation of energy and the consequent uniform a priori bound on the H^1-norm of the solution, global well-posedness holds true.

- For $N = 3$, if the nonlinearity is focusing (i.e. its coefficient is negative), then there exist initial data for which the L^2-norm of the gradient of the solution blows up in finite time.
- In the case $N = 2$ and focusing nonlinearity, we have
  - if $\|\psi_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the unique positive radial solution to $\Delta Q + Q^3 - Q = 0$, then the solution exists globally [36];
  - for initial data $\|\psi_0\|_{L^2} \geq \|Q\|_{L^2}$ there is possible occurrence of blow-up.

Unfortunately, in the case of system (1) the picture is not complete and we can give only a partial answer.

Theorem 5 ([17])**.** *Let $N \leq 3$ and set $\beta = \max\{2(-\beta_{11})^+, 2(-\beta_{22})^+, (-\beta_{12})^+\}$. Then there exists a global-in-time solution to in the following cases:*

1. $\beta_{11}, \beta_{22}, \beta_{12} \geq 0$ *or* $\beta_{12}^2 < \beta_{11}\beta_{22}$ *with* $\beta_{11} \geq 0$
2. $N = 1$
3. $N = 2$ *and* $M(0) < 2/(C_2\beta)$, *if* $\min\{\beta_{11}, \beta_{22}, \beta_{12}\} < 0$
4. $N = 3$, $\|\nabla\Psi(0)\|_2^2 \leq 2(E(0) + |\lambda|M(0))$, *and* $M(0)(E(0) + |\lambda|M(0)) < \frac{8}{27C_3^2\beta^2}$, *if* $\min\{\beta_{11}, \beta_{22}, \beta_{12}\} < 0$

Proof. We show global existence for $\beta_{12}^2 < \beta_{11}\beta_{22}$ with $\beta_{11} > 0$. It follows that

$$\int_{\mathbb{R}^N} \big(\frac{\beta_{11}}{2}|\psi_2|^4 + \beta_{12}|\psi_1|^2|\psi_2|^2 + \frac{\beta_{22}}{2}|\psi_2|^2\big)(x,t)dx \geq 0$$

is always positive, thus using the energy conservation we get uniform bounds on the Σ-norm of the solution Ψ.

$$\frac{1}{2}\|\nabla\Psi(t)\|_{L^2}^2 + \frac{\gamma^2}{2}\||\cdot|\Psi\|_{L^2}^2 \leq E(t) - 2\lambda\int_{\mathbb{R}^N} \Re(\psi_1^*\psi_2)(x,t)dx \leq E(t) + |\lambda|M(t) = E(0) + |\lambda|M(0).$$

For all other cases see [17]. □

In the next Theorem we show the cases in which there is possible occurrence of blow-up in finite time. We describe two situations, the first one in which the nonlinearity is negative definite (see below). In this case we apply a method introduced by Carles [6] for focusing NLS equations with a confining potential, by using a modified energy functional (for more discussions about this modified energy and its interpretation see [6]). In the second case we assume that at least one of the coefficients β_{ij}'s is focusing and that the energy is negative enough so that conditions (ii) or (iii) below are fulfilled. In this case we apply the method by Glassey [16] using virial identities.

Remark 6. *Note that the following sufficient condition for blow-up is valid also in the supercritical case $N = 3$, whereas in [17] the authors consider only the mass-critical case $N = 2$.*

Theorem 7. *Let $\Psi \in \mathcal{C}([0, T_{max}); \Sigma(\mathbb{R}^N))$ be the solution to (1) as in Corollary 4, and let us define the virial potential*

$$I(t) = \int_{\mathbb{R}^N} |x|^2|\Psi(x,t)|^2dx.$$

Let us assume $N \geq 2$ and one of the following conditions is satisfied

(i) *the nonlinearity is* negative definite, *i.e. we either have $\beta_{12}^2 - \beta_{11}\beta_{22} < 0$ with $\beta_{11} < 0$, or $\beta_{11}, \beta_{12}, \beta_{22} < 0$, and we also assume that*

$$E(0) + \frac{|\lambda|}{2} M(0) < \frac{\gamma^2}{2} I(0);$$

(ii) $\min\{\beta_{11}, \beta_{22}, \beta_{12}\} < 0$

$$\frac{2N}{N+2} \left(E(0) + |\lambda| M(0)\right) < \frac{\gamma^2}{2} I(0);$$

(iii) $\min\{\beta_{11}, \beta_{22}, \beta_{12}\} < 0$; $I'(0) < 0$ *and*

$$\frac{2N}{N+2} \left(E(0) + |\lambda| M(0)\right) < -\frac{\gamma}{\sqrt{2+N}} I'(0).$$

Then the solutions blows-up at a finite time, i.e. $\exists\, 0 < T^* < \infty$, such that

$$\lim_{t \to T^*} \|\nabla \Psi(t)\|_{L^2} = \infty.$$

Proof. We first consider case (i). Analogously to [6] we introduce the following functional

$$\begin{aligned} E_1(t) = {} & \cos^2(\gamma t) \int_{\mathbb{R}^N} \big(\frac{1}{2}|\nabla \Psi|^2 + \frac{1}{2}\Psi^* \tilde{B}[\Psi]\Psi + 2\lambda \Re(\psi_1^* \psi_2)\big)(t,x)dx \\ & + \sin^2(\gamma t) \int_{\mathbb{R}^N} \frac{\gamma^2}{2} |x|^2 |\Psi(t,x)|^2 dx + \frac{\gamma}{2}\sin(2\gamma t) \int_{\mathbb{R}^N} x \cdot J(t,x) dx \\ & + |\lambda| \frac{\cos(2\gamma t)}{2} \int_{\mathbb{R}^N} |\Psi(t,x)|^2 dx, \end{aligned}$$

where $J(t,x)$ is the current density defined by

$$J(x,t) = \Im(\psi_1^* \nabla \psi_1 + \psi_2^* \nabla \psi_2).$$

Let us first notice that the condition $E(0) + \frac{|\lambda|}{2} M(0) < \frac{\gamma^2}{2} I(0)$ implies $E_1(0) < 0$. By computing the time derivative of E_1 we obtain

$$\frac{d}{dt} E_1(t) = \gamma \sin(2\gamma t) \frac{N-2}{4} \int_{\mathbb{R}^N} \Big(\Psi \tilde{B}[\Psi]\Psi\Big)(t,x)dx + |\lambda|\gamma \sin(2\gamma t) \int_{\mathbb{R}^N} \big(2\,\mathrm{sign}(\lambda)\Re(\psi_1^*\psi_2) - |\Psi|^2\big)(t,x)dx.$$

By the assumptions on the coefficients β_{ij}'s we have $\Psi \tilde{B}[\Psi]\Psi < 0$ for any $\Psi \neq 0$, we thus infer

$$\frac{d}{dt} E_1(t) \leq 0, \quad \forall\, t \in \left[0, \frac{\pi}{2\gamma}\right]. \tag{14}$$

On the other hand, we see that if Ψ exists in $[0, \frac{\pi}{2\gamma}]$, then

$$E_1\left(\frac{\pi}{2\gamma}\right) = \frac{\gamma^2}{2} \int |x|^2 |\Psi(x,t)|^2 dx \geq 0,$$

contradicting $E_1(0) < 0$ and (14). Thus, there exists $T^* \leq \frac{\pi}{2\gamma}$ such that

$$\lim_{t\to T^*} \|\nabla\Psi(t)\|_{L^2} = \infty.$$

For cases (ii) and (iii), we use the virial identities. By similar calculations as above we have

$$\begin{aligned} I'(t) =& 2\int_{\mathbb{R}^N} x\cdot J(x,t)dx \\ I''(t) =& \int_{\mathbb{R}^N} \left(2|\nabla\Psi|^2 + N\Psi^*\tilde{B}[\Psi]\Psi - 2\gamma^2|x|^2|\Psi|^2\right)(t,x)dx. \end{aligned}$$

We write

$$I''(t) = 2NE(t) + 2(2-N)\int_{\mathbb{R}^N} |\nabla\Psi(x,t)|^2dx - (2+N)\gamma^2 I(t) - 4N\lambda\int_{\mathbb{R}^N} \Re(\psi_1^*\psi_2)(t,x)dx.$$

From the conservation of energy and mass we then obtain

$$I''(t) = -(2+N)\gamma^2 I(t) + 2N(E(0) + |\lambda|M(0)) + R(t),$$

where

$$R(t) = 2(2-N)\int_{\mathbb{R}^N} |\nabla\Psi(t,x)|^2dx - 4N\lambda\int_{\mathbb{R}^N} \Re(\psi_1^*\psi_2)(t,x)dx - 2N|\lambda|M(t) \leq 0.$$

The solution of the above differential equation is given by

$$\begin{aligned} I(t) =& \cos(\sqrt{2+N}\gamma t)I(0) + \frac{1}{\sqrt{2+N}\gamma}\sin(\sqrt{2+N}\gamma t)I'(0) \\ &+ \frac{2N}{(2+N)\gamma^2}(E(0) + |\lambda|M(0))(1-\cos(\sqrt{2+N}\gamma t)) \\ &+ \frac{1}{\sqrt{2+N}\gamma}\int_0^t \sin(\sqrt{2+N}\gamma(t-s))R(s)ds \\ \leq& \cos(\sqrt{2+N}\gamma t)I(0) + \frac{1}{\sqrt{2+N}\gamma}\sin(\sqrt{2+N}\gamma t)I'(0) \\ &+ \frac{2N}{(2+N)\gamma^2}(E(0) + |\lambda|M(0))(1-\cos(\sqrt{2+N}\gamma t)). \end{aligned}$$

We then notice that, if (ii) or (iii) are satisfied, then $I(t)$ will eventually become negative, giving a contradiction. The unboundedness of the gradient then follows from the conservation of mass and the uncertainty inequality

$$\|\Psi(t)\|_{L^2}^2 \leq \frac{2}{N}\||\cdot|\Psi(t)\|_{L^2}\|\nabla\Psi(t)\|_{L^2} \lesssim \sqrt{I(t)}\|\nabla\Psi(t)\|_{L^2}.$$

$\square$

3. The transformed system

As we showed in Theorem 5 the picture about global existence or possible blow-up of solutions to (1) is not complete. In order to get a better understanding, at least in the case when $|\lambda|$ becomes large, we are going to study the asymptotic behavior of solutions to (1) and the limiting system. In this Section we transform the system (1) into a similar one, which is more suitable to handle in order to study the asymptotic limit. Let us consider the linear part of (1),

$$i\partial_t\Psi = -\frac{1}{2}\Delta\Psi + \frac{\gamma^2}{2}|x|^2\Psi + A\Psi. \tag{15}$$

If $\lambda = 0$ (i.e. $A = 0$), then the two equations in (15) decouple and they evolve independently through the Hamiltonian $H := -\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2$ and we have $\Psi(t) = S_0(t)\Psi_0$. In the case $\lambda \neq 0$ the two equations are coupled, and thus

$$\Psi(t) = S_\lambda(t)\Psi_0,$$

where $S_\lambda(t) = S_0(t)\Omega_\lambda(t) = e^{-itH}\Omega_\lambda(t)$, and

$$\Omega_\lambda(t) := e^{-itA} = \begin{pmatrix} \cos(\lambda t) & -i\sin(\lambda t) \\ -i\sin(\lambda t) & \cos(\lambda t) \end{pmatrix}.$$

It is straightforward to see that $\Omega_\lambda(t)$ satisfies the following properties

- $\Omega_\lambda(t)^* = \Omega_\lambda(-t)$ for $\forall\, t \in \mathbb{R}$
- $\Omega_\lambda(t_1)\Omega_\lambda(t_2) = \Omega_\lambda(t_1 + t_2)$ for every $t_1, t_2 \in \mathbb{R}$.

System (1) can now be written in the following integral formulation

$$\Psi(t) = S_\lambda(t)\Psi_0 - i\int_0^t S_\lambda(t-s)\tilde{B}[\Psi]\Psi(s)ds. \tag{16}$$

To study the asymptotic limit when $|\lambda| \to \infty$, we want to cancel out the oscillations in the homogeneous part. We write

$$\Omega_\lambda(-t)\Psi(t) = S_0(t)\Psi_0 - i\int_0^t S_0(t-s)\Omega_\lambda(-s)\tilde{B}[\Psi]\Omega_\lambda(s)\Omega_\lambda(-s)\Psi(s)ds. \tag{17}$$

Now we define

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to be the matrix having v_1, v_2 as columns, so that $\Omega_\lambda(t)v_1 = e^{-i\lambda t}v_1$, $\Omega_\lambda(t)v_2 = e^{i\lambda t}v_2$, and

$$\Omega_\lambda(t)V = V\begin{pmatrix} e^{-i\lambda t} & \\ & e^{i\lambda t} \end{pmatrix}. \tag{18}$$

By using v_1, v_2 we define

$$\begin{aligned}\phi_1(t) :=&\langle v_1, \Omega_\lambda(-t)\Psi(t)\rangle = e^{i\lambda t}\langle v_1, \Psi(t)\rangle\\ \phi_2(t) :=&\langle v_2, \Omega_\lambda(-t)\Psi(t)\rangle = e^{-i\lambda t}\langle v_2, \Psi(t)\rangle,\end{aligned}$$

so that $\Psi(t) = e^{-i\lambda t}\phi_1(t)v_1 + e^{i\lambda t}\phi_2(t)v_2$ and

$$\Omega(-t)\Psi(t) = \phi_1(t)v_1 + \phi_2(t)v_2.$$

We want to find the system of NLS equations satisfied by $\Phi = (\phi_1, \phi_2)^t$. From formula (17) we can see that

$$\Phi(t) = S_0(t)\Phi_0 - i\int_0^t S_0(t-s)V^t\Omega_\lambda(-s)\tilde{B}[\Psi]\Omega_\lambda(s)V\Phi(s)ds, \tag{19}$$

where $\Phi_0 := V^t\Psi_0$. Now we notice that $\tilde{B}[\Psi] = \begin{pmatrix} \psi_1 & \\ & \psi_2 \end{pmatrix} B \begin{pmatrix} \psi_1^* & \\ & \psi_2^* \end{pmatrix}$, and we may write

$$\begin{pmatrix} \psi_1 & \\ & \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(e^{-i\lambda t}\phi_1(t)\mathbb{I} + e^{i\lambda t}\phi_2(t)\mathbb{I}_-\right), \tag{20}$$

where $\mathbb{I}_- = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

We want to compute the nonlinear potential matrix $V^t\Omega_\lambda(-s)\tilde{B}[\Psi]\Omega_\lambda(s)V =: \hat{B}[\Phi]$ in formula (19). By using (18) and (20) we see it equals

$$\frac{1}{2}\begin{pmatrix} e^{i\lambda s} & \\ & e^{-i\lambda s} \end{pmatrix} V^t\left(B|\phi_1|^2 + \mathbb{I}_-B\mathbb{I}_-|\phi_2|^2 + e^{-2i\lambda s}B\mathbb{I}_-\phi_1\phi_2^* + e^{2i\lambda s}\mathbb{I}_-B\phi_1^*\phi_2\right)V\begin{pmatrix} e^{-i\lambda s} & \\ & e^{i\lambda s} \end{pmatrix}.$$

After some tedious calculations, and by rearranging a bit the terms we write

$$\begin{aligned}\hat{B}[\Phi] =&\frac{1}{4}\left[(\beta_{11} + 2\beta_{12} + \beta_{22})\begin{pmatrix} |\phi_1|^2 & \phi_1\phi_2^* \\ \phi_1^*\phi_2 & |\phi_2|^2 \end{pmatrix} + (\beta_{11} - 2\beta_{12} + \beta_{22})\begin{pmatrix} |\phi_2|^2 & \\ & |\phi_1|^2 \end{pmatrix}\right]\\ &+ \frac{1}{4}(\beta_{11} - \beta_{22})\left[\mathbb{I}\left(e^{-2i\lambda t}\phi_1\phi_2^* + e^{2i\lambda t}\phi_1^*\phi_2\right) + \begin{pmatrix} & e^{2i\lambda t} \\ e^{-2i\lambda t} & \end{pmatrix}(|\phi_1|^2 + |\phi_2|^2)\right]\\ &+ \frac{1}{4}(\beta_{11} - 2\beta_{12} + \beta_{22})\begin{pmatrix} & e^{4i\lambda t}\phi_1^*\phi_2 \\ e^{-4i\lambda t}\phi_1\phi_2^* & \end{pmatrix}\\ =&: \hat{B}^\infty[\Phi] + R^\lambda[\Phi],\end{aligned} \tag{21}$$

where we divided the autonomous part $\hat{B}^\infty[\Phi]$ from the non-autonomous one $R^\lambda[\Phi]$. Consequently, the equation for the transformed variable Φ becomes

$$i\partial_t\Phi = -\frac{1}{2}\Delta\Phi + \frac{\gamma^2}{2}|x|^2\Phi + \hat{B}[\Phi]\Phi, \tag{22}$$

with $\hat{B}$ defined above in (21). Let us notice that in (22) the term $\hat{B}^\infty[\Phi]\Phi$ can be also written in the following compact form

$$\frac{1}{4}\begin{pmatrix} (\beta_{11}+2\beta_{12}+\beta_{22})|\phi|^2 & 2(\beta_{11}+\beta_{22})\phi_1\phi_2^* \\ 2(\beta_{11}+\beta_{22})\phi_1^*\phi_2 & (\beta_{11}+2\beta_{12}+\beta_{22})|\phi_2|^2 \end{pmatrix}\Phi,$$

thus we may rename $\hat{B}^\infty[\Phi]$ to be the matrix in the expression above, i.e.

$$\hat{B}^\infty[\Phi] = \frac{1}{4}\begin{pmatrix} (\beta_{11}+2\beta_{12}+\beta_{22})|\phi|^2 & 2(\beta_{11}+\beta_{22})\phi_1\phi_2^* \\ 2(\beta_{11}+\beta_{22})\phi_1^*\phi_2 & (\beta_{11}+2\beta_{12}+\beta_{22})|\phi_2|^2 \end{pmatrix}. \tag{23}$$

It is straightforward to check that the nonlinear terms $\hat{B}^\infty, R^\lambda$ satisfy

$$|\hat{B}^\infty[F_1]F_1 - \hat{B}^\infty[F_2]F_2| \lesssim (|F_1|^2+|F_2|^2)|F_1-F_2|, \tag{24}$$

$$|\nabla\left(\hat{B}^\infty[F_1]F_1 - \hat{B}^\infty[F_2]F_2\right)| \lesssim (|F_1|^2+|F_2|^2)|\nabla(F_1-F_2)| \tag{25}$$

$$|\nabla^k(R^\lambda[F_1]F_1)| \lesssim |F_1|^2|\nabla^k F_1|, \quad k=0,1 \tag{26}$$

Remark 8. *Since the Σ-norms of Ψ and of Φ^λ are equal, all the results stated in Corollary 4 and Theorems 7 and 5 hold true also for Φ^λ.*

Formally, in the limit as $|\lambda|$ goes to infinity, we expect the non-autonomous part to cancel out, because of the oscillating coefficients in front of the nonlinearities which average to zero. We thus obtain the following system of coupled nonlinear Schrödinger equations

$$i\partial_t U = -\frac{1}{2}\Delta U + \frac{\gamma^2}{2}|x|^2 U + \hat{B}^\infty[U]U. \tag{27}$$

and the corresponding limiting energy and mass are given by

(28)

$$\begin{aligned}\hat{E}(t) &= \int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla U|^2 + \frac{\gamma^2}{2}|x|^2|U|^2 + \frac{1}{2}U^*\hat{B}^\infty[U]U\right)(x,t)dx \\ &= \int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla U|^2 + \frac{\gamma^2}{2}|x|^2|U|^2 + \frac{\beta_{11}+2\beta_{12}+\beta_{22}}{8}(|u_1|^4+|u_2|^4) + \frac{\beta_{11}+\beta_{22}}{2}|u_1|^2|u_2|^2\right)(x,t)dx, \\ \hat{M}_j(t) &= \int_{\mathbb{R}^N}|u_j(x,t)|^2dx, \text{ with } j=1,2.\end{aligned}$$

Remark 9. *Let us consider formula* (17) *again,*

$$\Omega_\lambda(-t)\Psi(t) = S_0(t)\Psi_0 - i\int_0^t S_0(t-s)\Omega_\lambda(-s)\tilde{B}[\Psi]\Omega_\lambda(s)\Omega_\lambda(-s)\Psi(s)ds.$$

If we define

$$\tilde{\Phi}(t) := \Omega_\lambda(-t)\Psi(t),$$

then we obtain the following system

$$\tilde{\Phi}(t) = S_0(t)\tilde{\Phi}_0 - i\int_0^t S_0(t-s)B_2[\tilde{\Phi}]\tilde{\Phi}(s)ds,$$

but now the nonlinear potential matrix $B_2[\tilde{\Phi}]$ is given by the more complicated expression

$$\begin{aligned}
B_2[\tilde{\Phi}] =& \frac{1}{4}(\beta_{11}+\beta_{22})|\tilde{\Phi}|^2 + \frac{\beta_{12}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(|\tilde{\phi}_1|^2 - |\tilde{\phi}_2|^2) \\
&+ \frac{1}{8}(\beta_{11}+2\beta_{12}+\beta_{22})\left[\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\tilde{\phi}_1\tilde{\phi}_2^* + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\phi_1^*\phi_2\right] \\
&+ e^{-2i\lambda t}\left\{\frac{1}{8}(\beta_{11}-\beta_{22})\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}(|\tilde{\phi}_1|^2 + |\tilde{\phi}_2|^2) + \frac{1}{4}(\beta_{11}-\beta_{22})\mathbb{I}\tilde{\phi}_1\tilde{\phi}_2^*\right\} \\
&+ e^{2i\lambda t}\left\{\frac{1}{8}(\beta_{11}-\beta_{22})\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}(|\tilde{\phi}_1|^2 + |\tilde{\phi}_2|^2) + \frac{1}{4}(\beta_{11}-\beta_{22})\mathbb{I}\tilde{\phi}_1^*\tilde{\phi}_2\right\} \\
&+ \frac{1}{8}(\beta_{11}-2\beta_{12}+\beta_{22})\left\{e^{-4i\lambda t}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\tilde{\phi}_1\tilde{\phi}_2^* + e^{4i\lambda t}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\tilde{\phi}_1^*\tilde{\phi}_2\right\}.
\end{aligned}$$

Furthermore, if we only consider the autonomous part, we see that even in the asymptotic limit we obtain an expression for the nonlinear potential matrix which is rather complicated.
On the other hand, if all the inter- and intra-species coefficients equal, i.e. $\beta_{11} = \beta_{12} = \beta_{22} \equiv \beta$, then the expression (and the analysis) simplifies considerably. Numerical studies using this transformation in the case of equal coefficients are performed in [13, 28]. *Nevertheless our aim in this paper is to consider a general choice for the coefficients β_{ij}'s, that is why we choose to transform the system in terms of Φ.*

4. Asymptotics for $\lambda \to \infty$

In this Section we prove the rigorous convergence of solutions to system (22), towards solutions to system (27), when the parameter $|\lambda|$ goes to infinity. As the reader can see it is heavily inspired by the paper by Cazenave, Scialom [10]; here a system of NLS equations is considered, and we also have to deal with the presence of a confining potential, but the strategy of proof follows the same steps as [10].
The main result of this Section is Theorem 11, but before stating it we first analyze system (27). By Proposition 3 and Theorem 5 we have the following

Proposition 10. *Let $U_0 \in \Sigma(\mathbb{R}^N)$. Then there exist $\delta = \delta(\|U_0\|_{\Sigma(\mathbb{R}^N)}) > 0$ and $U \in \mathcal{C}([0,\delta];\Sigma(\mathbb{R}^N))$ unique solution to system* (27). *Moreover,*

$$\|U\|_{L^\infty([0,\delta];\Sigma(\mathbb{R}^N))} \leq 2C\|U_0\|_{\Sigma(\mathbb{R}^N)}. \tag{29}$$

Furthermore, the solution U can be extended to a maximal interval $[0, S_{max})$, we have $U \in \mathcal{C}([0,S_{max});\Sigma(\mathbb{R}^N))$ and the usual blow-up alternative holds, i.e. if $S_{max} < \infty$, then $\|\nabla U(t)\|_{L^2} \to \infty$, as $t \nearrow S_{max}$.
Moreover, $S_{max} = \infty$, thus the solution U to (27) *is global-in-time in the following cases:*

- $\beta_{11}+\beta_{22} \geq 0$ *and* $\beta_{11}+2\beta_{12}+\beta_{22} \geq 0$
- $\beta_{11}+\beta_{22} < 0$ *and* $\frac{3}{2}|\beta_{11}+\beta_{22}| < \beta_{12}$

For any $0 < T < S_{max}$, (q, r) *admissible pair, we have*

$$U, \nabla U, |\cdot|U \in L^q([0,T]; L^r(\mathbb{R}^N)).$$

Finally, the three quantities defined in (28)*, namely the total energy and the mass of each component, are conserved in* $[0, S_{max})$*.*

Now we can state our main Theorem.

Theorem 11. *Let* $\Phi_0 \in \Sigma(\mathbb{R}^N)$*. For any* $\lambda \in \mathbb{R}$*, we denote by* Φ^λ *the unique maximal solution to* (22)*. Let* U *be the solution to* (27)*, with initial data* $U(0) = \Phi_0$*, in* $[0, S_{max})$*, where* $0 < S_{max} \leq \infty$*, as in Proposition 10.*

- *For any* $0 < T < S_{max}$*, the solution* Φ^λ *exists in* $[0, T]$ *provided* $|\lambda|$ *is sufficiently large.*
- *For any* $0 < T < S_{max}$*,* (q, r) *admissible pair, we have*

$$\|\Phi^\lambda - U\|_{L^q_t L^r_x} + \|\nabla(\Phi^\lambda - U)\|_{L^q_t L^r_x} + \||\cdot|(\Phi^\lambda - U)\|_{L^q_t L^r_x} \to 0, \tag{30}$$

as $|\lambda| \to \infty$*, where the* $L^q_t L^r_x$*−norms are taken in the space-time slab* $[0, T] \times \mathbb{R}^N$*. In particular, convergence holds in* $\mathcal{C}([0, T]; \Sigma(\mathbb{R}^N))$*.*

The idea of the proof can be described as follows; let us consider the equation satisfied by the difference $\Phi^\lambda - U$. By using the integral formulation we may write

$$\begin{aligned}\Phi^\lambda(t) - U(t) = &- i \int_0^t S_0(t-s) \left[\hat{B}^\infty[\Phi^\lambda]\Phi^\lambda - \hat{B}^\infty[U]U\right](s) ds \\ &- i \int_0^t S_0(t-s) R^\lambda[\Phi^\lambda]\Phi^\lambda(s) ds =: -iI_1 - iI_2,\end{aligned}$$

see (21), (23) for the definitions of $\hat{B}^\infty, R^\infty$.
The oscillating coefficients in R^λ converge weakly to zero, and since they appear inside the time integral in the Duhamel's formula, then I_2 converges (strongly) to zero. For the I_1 part, on the other hand, we can use a Lipschitz estimate for the nonlinearity $\hat{B}^\infty[F]F$ and close the convergence with a continuity argument.
To prove Theorem 11 above we proceed in two steps: first we prove that, as long as we have uniform bounds on Φ^λ on a space-time slab, we obtain the convergence. Then we prove that we indeed have such bounds on Φ^λ in $[0, T] \times \mathbb{R}^N$, with $0 < T < S_{max}$, provided $|\lambda|$ is sufficiently large.
We start by giving a technical Lemma which will be used to prove the convergence of I_2 to zero.

Lemma 12. *Let* $(\tilde{q}, \tilde{r})$ *be an admissible pair,* $0 < T < \infty$*, and let* $f \in L^{\tilde{q}'}([0, T]; L^{\tilde{r}'}(\mathbb{R}^N))$*. Then, for any admissible pair* (q, r)*, we have*

$$\|\int_0^t S_0(t-s) e^{i\lambda s} f(s) ds\|_{L^q([0,T]; L^r(\mathbb{R}^N))} \to 0, \quad as\ |\lambda| \to \infty. \tag{31}$$

Proof. First of all, let us notice that by Strichartz estimates we have

$$\|\int_0^t S_0(t-s)e^{i\lambda s}f(s)ds\|_{L^q([0,T];L^r(\mathbb{R}^N))}\lesssim\|f\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}.$$

Then, it is straightforward to see that by using a density argument it suffices to prove (31) for all $f\in\mathcal{C}^1([0,T];\mathcal{S}(\mathbb{R}^N))$. Let us consider the integral in (31), by integration by parts we obtain

$$\int_0^t e^{i\lambda s}S_0(t-s)f(s)ds=\frac{1}{i\lambda}\left(e^{i\lambda t}f(t)-S_0(t)f(0)\right)-\frac{1}{i\lambda}\int_0^t S_0(t-s)\left[\partial_s f(s)-Hf(s)\right]ds.$$

Thus again by Strichartz estimates we get

$$\|\int_0^t S_0(t-s)e^{i\lambda s}f(s)ds\|_{L_t^qL_x^r}\lesssim\frac{1}{|\lambda|}\left(\|f\|_{L_t^qL_x^r}+\|f(0)\|_{L^2}+\|\partial_s f-Hf\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}\right),$$

and the right hand side goes to zero as $|\lambda|\to\infty$. $\square$

Remark 13. *In* (31) *we proved the convergence for the term with* $e^{i\lambda s}$, *but this clearly holds for any other periodic function which has zero average, in particular it also holds for all the coefficients appearing in front of the nonlinearities in* R^λ, *see* (21).

The next Proposition shows that, as long as we have uniform bounds on the $L_t^\infty\Sigma_x-$norm of Φ^λ on a space-time slab, then we can prove the convergence of Φ^λ towards U in that space-time slab.

Proposition 14. *Let* $\Phi_0\in\Sigma(\mathbb{R}^N)$ *and let* Φ^λ *denote the maximal solution of* (22). *Let* U *be the maximal solution of* (27), *defined on* $[0,S_{max})$. *Let* $0<\ell<S_{max}$ *and assume that* Φ^λ *exists on* $[0,\ell]\times\mathbb{R}^N$ *and that*

$$\limsup_{|\lambda|\to\infty}\|\Phi^\lambda\|_{L^\infty([0,\ell];\Sigma(\mathbb{R}^N))}<\infty. \tag{32}$$

Then we have

$$\lim_{|\lambda|\to\infty}\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}(\Phi^\lambda-U)\|_{L^q([0,\ell];L^r(\mathbb{R}^N))}=0 \tag{33}$$

for any admissible pairs (q,r). *In particular* $\Phi^\lambda\to U$ *in* $L^\infty([0,\ell];\Sigma(\mathbb{R}^N))$.

Proof. By hypothesis (32), we can take L large enough such that

$$\sup_{|\lambda|\geq L}\|\Phi^\lambda\|_{L^\infty([0,\ell];\Sigma(\mathbb{R}^N))}<\infty.$$

By using Strichartz estimates similarly to the proof of Proposition 3 we see that from the bound above we also infer

$$\sup_{|\lambda|\geq L}\sup_{(q,r)}\|\begin{pmatrix}1\\ \nabla\\ |\cdot|\end{pmatrix}\Phi^\lambda\|_{L_t^qL_x^r([0,\ell]\times\mathbb{R}^N)}<\infty, \tag{34}$$

where the second supremum is taken over all admissible pairs (q, r). If Φ^λ is a solution to (22) and U to (27), then we have

$$\begin{aligned}\Phi^\lambda(t) - U(t) = &- i \int_0^t S_0(t-s)\left(\hat{B}^\infty[\Phi^\lambda]\Phi^\lambda - \hat{B}^\infty[U]U\right)(s)ds \\ &- i \int_0^t S_0(t-s)R^\lambda[\Phi^\lambda]\Phi^\lambda(s)ds =: -iI_1 - iI_2,\end{aligned}$$

where $\hat{B}^\infty, R^\lambda$ are defined in (21), (23). We first consider I_1; by using Strichartz estimates, inequality (24), Hölder's inequality and then (32), we have

$$\begin{aligned}\|I_1\|_{L^q([0,\ell];L^r(\mathbb{R}^N))} \leq& C(\|\Phi^\lambda\|^2_{L^\infty([0,\ell];L^4(\mathbb{R}^N))} + \|U\|^2_{L^\infty([0,\ell];L^4(\mathbb{R}^N))})\|\Phi^\lambda - U\|_{L^{\frac{8}{8-N}}([0,\ell];L^4(\mathbb{R}^N))} \\ \leq& C\|\Phi^\lambda - U\|_{L^{\frac{8}{8-N}}([0,\ell];L^4(\mathbb{R}^N))}.\end{aligned}$$

For I_2 we use Lemma 12, indeed by Sobolev embedding we have

$$\|R^\lambda[\Phi^\lambda]\Phi^\lambda\|_{L_t^{\frac{8}{8-N}}L_x^{4/3}([0,\ell]\times\mathbb{R}^N)} \lesssim \|\Phi^\lambda\|^3_{L_t^\infty L_x^4} \lesssim \|\Phi^\lambda\|^3_{L_t^\infty H_x^1},$$

and consequently

$$\|\int_0^t S_0(t-s)R^\lambda[\Phi^\lambda]\Phi^\lambda(s)ds\|_{L_t^qL_x^r([0,\ell]\times\mathbb{R}^N)}$$

converges to zero as $|\lambda| \to 0$, for any admissible pair (q, r). By choosing $(q, r) = (\frac{8}{N}, 4)$, we have

$$\|\Phi^\lambda - U\|_{L_t^{8/N}L_x^4([0,\ell]\times\mathbb{R}^N)} \leq \varepsilon + C\|\Phi^\lambda - U\|_{L_t^{\frac{8}{8-N}}L_x^4([0,\ell]\times\mathbb{R}^N)}.$$

A standard continuity argument gives us

$$\|\Phi^\lambda - U\|_{L_t^{8/N}L_x^4([0,\ell]\times\mathbb{R}^N)} \leq C\varepsilon,$$

which proves the convergence $\Phi^\lambda \to U$ in $L_t^{8/N}L_x^4([0,\ell]\times\mathbb{R}^N)$. The same holds for any admissible pair (q, r), since

$$\|\Phi^\lambda - U\|_{L_t^qL_x^r} \leq \|I_1\|_{L_t^qL_x^r} + \|I_2\|_{L_t^qL_x^r} \leq \varepsilon + C\|\Phi^\lambda - U\|_{L_t^{\frac{8}{8-N}}L_x^4} \leq \varepsilon + C\|\Phi^\lambda - U\|_{L_t^{8/N}L_x^4},$$

where in the last inequality we used Hölder in time. Consequently $\Phi^\lambda \to U$ in $L_t^qL_x^r([0,\ell]\times\mathbb{R}^N)$ for any admissible pair. We now prove the convergence of $\nabla\Phi^\lambda$ and $|\cdot|\Phi^\lambda$ in the same spaces. Again we consider the difference

$$\nabla(\Phi^\lambda - U) + x(\Phi^\lambda - U) =: -iI_1 - iI_2 - iI_3,$$

where, by using the commutator relations of the Hamiltonian H with x and ∇, see (9), we obtain

$$\begin{aligned}
I_1 &:= \int_0^t S_0(t-s)\nabla\left(\hat{B}^\infty[\Phi^\lambda]\Phi^\lambda - \hat{B}^\infty[U]U\right)(s)ds,\\
I_2 &:= \int_0^t S_0(t-s)x\left(\hat{B}^\infty[\Phi^\lambda]\Phi^\lambda - \hat{B}^\infty[U]U\right)(s)ds,\\
I_3 &:= \int_0^t S_0(t-s)\left(\nabla(R^\lambda[\Phi^\lambda]\Phi^\lambda) + xR^\lambda[\Phi^\lambda]\Phi^\lambda\right)(s)ds.
\end{aligned}$$

By using Strichartz estimates, inequality (25) and Sobolev embedding, we have

$$\|I_1\|_{L_t^{8/N}L_x^4([0,\ell]\times\mathbb{R}^N)} \leq C\left(\|\Phi^\lambda\|^2_{L_t^\infty H_x^1} + \|U\|^2_{L_t^\infty H_x^1}\right)\|\nabla(\Phi^\lambda - U)\|_{L_t^{\frac{8}{8-N}}L_x^4}.$$

Analogously, for I_2 we have

$$\|I_2\|_{L_t^{8/N}L_x^4([0,\ell]\times\mathbb{R}^N)} \leq C\left(\|\Phi^\lambda\|^2_{L_t^\infty H_x^1} + \|U\|^2_{L_t^\infty H_x^1}\right)\||\cdot|(\Phi^\lambda - U)\|_{L_t^{\frac{8}{8-N}}L_x^4}.$$

Finally, for I_3 we use inequality (26), Sobolev embedding and (34) to estimate

$$\|\nabla(R^\lambda[\Phi^\lambda]\Phi^\lambda)+|\cdot|R^\lambda[\Phi^\lambda]\Phi^\lambda\|_{L_t^{\frac{8}{8-N}}L_x^{4/3}} \leq C\|\Phi^\lambda\|^2_{L_t^\infty H_x^1}\left(\|\nabla\Phi^\lambda\|_{L_t^{8/N}L_x^4} + \||\cdot|\Phi^\lambda\|_{L_t^{8/N}L_t^4}\right) < \infty.$$

Hence we can apply Lemma 12 and infer that

$$\|I_3\|_{L_t^qL_x^r([0,\ell]\times\mathbb{R}^N)} \to 0, \text{as } |\lambda|\to\infty,$$

for any (q,r) admissible pair. By resuming, we have obtained

$$\|\nabla(\Phi^\lambda - U)\|_{L_t^{8/N}L_x^4} + \||\cdot|(\Phi^\lambda - U)\|_{L_t^{8/N}L_x^4} \leq \varepsilon + C\left(\|\nabla(\Phi^\lambda - U)\|_{L_t^{8/N}L_x^4} + \||\cdot|(\Phi^\lambda - U)\|_{L_t^{8/N}L_x^4}\right).$$

As before, a standard continuity argument gives us

$$\|\nabla(\Phi^\lambda - U)\|_{L_t^{8/N}L_x^4} + \||\cdot|(\Phi^\lambda - U)\|_{L_t^{8/N}L_x^4} \leq C\varepsilon,$$

which shows the convergence in $L_t^{8/N}L_x^4$ as $|\lambda|\to\infty$. The convergence in $L_t^qL_x^r([0,\ell]\times\mathbb{R}^N)$ for any (q,r) admissible pair then follows as before, by using Strichartz estimates. $\square$

The Lemma above states that, as long as the $L^\infty([0,\ell];\Sigma(\mathbb{R}^N))-$norm of the family $\{\Phi^\lambda\}$ of solutions for the Cauchy problem related to (22) is definitely bounded (as $|\lambda|\to\infty$) in a time interval $[0,\ell]$, then the family converges in to U solution to (27) with the same initial datum, in every Strichartz space.

It thus remains to prove that given any time interval $[0,T]$ with $0 < T < S_{max}$, the $L^\infty([0,T];\Sigma(\mathbb{R}^N))-$norm of Φ^λ is indeed uniformly bounded, if we take $|\lambda|$ sufficiently large.

For this purpose, let then T be a positive time strictly less than the maximal existence time for the solution U of (27), i.e. $0 < T < S_{max}$, and let us fix the constant

$$M := 2C\|U\|_{L^\infty([0,T];\Sigma(\mathbb{R}^N))},$$

where C is the constant in (29). Let $\delta = \delta(M)$ be the constant in Proposition 3. Then Φ^λ exists in $[0,\delta]$ for all λ and furthermore

$$\sup_{\lambda\in\mathbb{R}} \|\Phi^\lambda\|_{L^\infty((0,\delta);\Sigma(\mathbb{R}^N))} \leq 2C\|\Phi_0\|_\Sigma,$$

by (12). Now, let $0 < \ell \leq T$ be such that Φ^λ exists in $[0,\ell]$, and that we have

$$\limsup_{|\lambda|\to\infty} \|\Phi^\lambda\|_{L^\infty((0,\ell);\Sigma(\mathbb{R}^N))} < \infty.$$

Notice that by the inequality above we see that we can always choose $\ell = \delta$. Thus, by Proposition 14 we have

$$\lim_{|\lambda|\to\infty} \left(\|\Phi^\lambda - U\|_{L^q((0,\ell);L^r(\mathbb{R}^N))} + \|\nabla(\Phi^\lambda - U)\|_{L^q((0,\ell);L^r(\mathbb{R}^N))} + \||\cdot|(\Phi^\lambda - U)\|_{L^q((0,\ell);L^r(\mathbb{R}^N))}\right) = 0$$

for all admissible pairs (q,r). In particular we have

$$\lim_{|\lambda|\to\infty} \|\Phi^\lambda(\ell) - U(\ell)\|_{\Sigma(\mathbb{R}^N)} = 0.$$

This implies that

$$\sup_{|\lambda|\geq\Lambda} \|\Phi^\lambda(\ell)\|_{\Sigma(\mathbb{R}^N)} \leq M,$$

for some $\Lambda > 0$ sufficiently large.
We can thus repeat the same argument, starting at time $t = \ell$. Consequently we have the solution Φ^λ exists in $[0,\delta+\ell]$ and we have a uniform bound for the $\Sigma-$ norm of Φ^λ,

$$\sup_{|\lambda|\geq\Lambda} \|\Phi^\lambda\|_{L^\infty((0,\delta+\ell);\Sigma(\mathbb{R}^N))} \leq 2CM.$$

Thus we can apply again Lemma 14 in the time interval $[\ell,\delta+\ell]$ and obtain the convergence of Φ^λ to U. Again, because of the convergence we have

$$\lim_{|\lambda|\to\infty} \|\Phi^\lambda(\delta+\ell) - U(\delta+\ell)\|_\Sigma = 0,$$

and in particular

$$\sup_{|\lambda|\geq\Lambda'} \|\Phi^\lambda(\delta+\ell)\|_{\Sigma(\mathbb{R}^N)} \leq M.$$

Thus we repeat this argument to prove the result in the whole time interval $[0,T]$.

Acknowledgements

The work of the second author has been supported by the Hertha-Firnberg Program of the Austrian Science Fund (FWF), grant T402-N13.

Appendix A. Simple example

To illustrate the application of the presented theory we give a very simple example. Let us consider the case of one focusing and one defocusing nonlinearity, where the interspecific scattering length is zero. Thus we consider the following system in $N = 2$:

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi + \frac{\gamma^2}{2}|x|^2\Psi + \tilde{B}[\Psi]\Psi + A\Psi, \tag{35}$$

with A as defined in (4), and

$$\tilde{B}[\Psi] = \begin{pmatrix} -\beta|\psi_1|^2 & 0 \\ 0 & \beta|\psi_2|^2 \end{pmatrix} \quad \beta > 0.$$

We choose initial data such that the condition (3) in Theorem 5 is not satisfied, thus the initial mass M(0) is not too small, and none of the conditions (i)-(iii) of Theorem 7 is satisfied. In this way, we are actually not able to say if the solution may blow-up or exist globally.

Remark 15. *If $\lambda = 0$ the system is decoupled and we have two scalar nonlinear Schrödinger equations, for which we know that ψ_2 exists globally, and ψ_1 may blow up in finite time for initial mass $\|\psi_{1,0}\|_{L^2(\mathbb{R}^2)} \geq \|Q\|_{L^2(\mathbb{R}^N)}$ with Q being the unique positive radial solution to $\Delta Q + Q^3 - Q = 0$.*

Performing the asymptotics for $|\lambda| \to \infty$ we observe that Ψ converges to U solution to the linear Schrödinger system:

$$i\partial_t U = -\frac{1}{2}\Delta U + \frac{\gamma^2}{2}|x|^2 U.$$

Applying Theorem 11 we can say that the solution Ψ of (35) exists in $[0, T]$, for any $T < \infty$, provided that λ is taken sufficiently large.

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